

Translational tilings by a polytope, with multiplicity

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Abstract

We study the problem of covering \mathbb{R}^d by overlapping translates of a convex body P , such that almost every point of \mathbb{R}^d is covered exactly k times. Such a covering of Euclidean space by translations is called a k -tiling. The investigation of tilings (i.e. 1-tilings in this context) by translations began with the work of Fedorov [3] and Minkowski [11]. Here we extend the investigations of Minkowski to k -tilings by proving that if a convex body k -tiles \mathbb{R}^d by translations, then it is centrally symmetric, and its facets are also centrally symmetric. These are the analogues of Minkowski's conditions for 1-tiling polytopes. Conversely, in the case that P is a rational polytope, we also prove that if P is centrally symmetric and has centrally symmetric facets, then P must k -tile \mathbb{R}^d for some positive integer k .

1 Introduction

Suppose we are given a convex object P , and a multiset of discrete translation vectors Λ . We wish to cover all of \mathbb{R}^d by translating P using the translation vectors in Λ , such that each point $x \in \mathbb{R}^d$ is covered exactly k times. Along the boundary points of P there may be some technical lower-dimensional problems, but if we require that each point which does not lie on the boundary of any translate of P to be covered exactly k times, then we call such a covering of \mathbb{R}^d a k -tiling. The traditional field of tilings of Euclidean space by translates of a single convex object P has a long and rich history. The usual notion of a tiling by translations is thus equivalent to the notion of a 1-tiling. The reader is invited to consult the books by Alexandrov [1] and Gruber [4] for a nice overview of the problem of tiling space with translates of one convex body.

Tilings of \mathbb{R}^d by translations of a single object have been extensively studied from as early as 1881 [3], by the mathematical crystallographer Fedorov, and are an active research area today. For example, translational tilings of sets on the real line have been studied in the 90's by Lagarias and Wang [8]. There is also a beautiful recent survey article on tilings in various different mathematical contexts, by Kolountzakis and Matolcsi [7].

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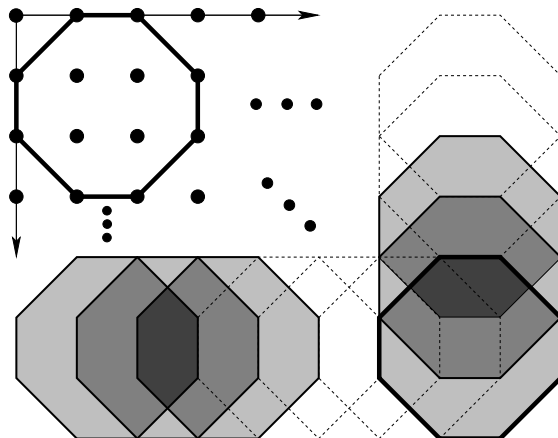


Figure 1: An octagon that 7-tiles, but does not 1-tile. Here, each point in the interior of the octagon is covered exactly 7 times, once we translate the octagon by all of the integer translation vectors.

We first note that if we have any k -tiling by a convex object P , then it is an elementary fact that the convex body P must be a polytope, and we may therefore assume henceforth that any convex object P that k -tiles is a polytope.

Minkowski [11] has shown that if a convex body P tiles \mathbb{R}^d by a lattice, then it follows that P is a centrally symmetric polytope, with centrally symmetric facets. Venkov [13] and McMullen [9] proved that if a convex body P tiles \mathbb{R}^d by translation, then for each of its codimension two faces F there are either four or six faces which are translates of F .

Here we find analogues of the necessary Minkowski conditions in the case of general k -tilings, for any integer k (see the main Theorem 1.1 below).

Despite the beautiful characterization of 1-tilers, given collectively by Minkowski, Venkov, and McMullen, there is still no known complete classification of polytopes that admit a k -tiling, even in two dimensions. However, it is known that in \mathbb{R}^2 , every k -tiling convex body has to be a centrally symmetric polygon. Also, there exists a characterization by Bolle [14] of all *lattice* k -tilings of convex bodies in \mathbb{R}^2 . Kolountzakis [6] proved that every k -tiling of \mathbb{R}^2 by a convex polygon P , which is not a parallelogram, is a k -tiling with a finite union of two-dimensional lattices.

A **parallelotope** is, by definition, a convex polytope that tiles (i.e. 1-tiles) \mathbb{R}^d facet-to-facet, with a lattice. That is, its multiset of discrete translation vectors Λ is in fact given by a lattice in this case. It was proved by McMullen that if a polytope tiles \mathbb{R}^d with a discrete multiset of translations Λ , then it must also admit a facet-to-facet tiling with a *lattice*. In other words, McMullen showed that every 1-tiler must be a parallelotope. A very active area of current research deals with the “Voronoi conjecture”, which

A **zonotope** in \mathbb{R}^d is a polytope which can be represented as a Minkowski sum of finitely many line segments. Equivalently, a zonotope is a polytope in \mathbb{R}^d with the property that all of its k -dimensional faces are centrally symmetric, for all $1 \leq k \leq d$. For example, the zonotopes

in \mathbb{R}^2 are the centrally symmetric polygons. A third equivalent definition for a zonotope is that it is the projection of a d -dimensional cube, for some d . For a good reference regarding these equivalences, and more about polytopes, see the book by G. Ziegler [15].

It is clear that not all zonotopes are parallelotopes, an easy example being furnished by the octagon (see fig.refoctagon) in two dimensions, which clearly does not tile by a lattice of translation vectors; conversely, not all parallelotopes are zonotopes, as evidenced by the example of the 24-cell given below. In fact, McMullen has given a beautiful characterization of those parallelotopes which are zonotopes, in terms of unimodular systems (see [10] for more details).

Very little is known about the precise classification of polytopes which k -tile \mathbb{R}^d by translations. We outline some specific open questions in the last section that pertain to the current state of affairs along these lines. (see [4], pages 463-479 for more details about 1-tiling polytopes and some open problems).

We can now state the main result of this paper.

Theorem 1.1. *If a convex polytope k -tiles \mathbb{R}^d by translations, then it is centrally symmetric and its facets are centrally symmetric.*

We note that in \mathbb{R}^3 these two conditions are enough for a convex body to necessarily be a zonotope. However, in dimension 4 this is no longer the case. A counterexample is furnished by the 24-cell, which is a polytope in \mathbb{R}^4 which 1-tiles \mathbb{R}^4 , is centrally symmetric, has centrally symmetric facets, but is not a zonotope because it has 2-dimensional faces that are triangles. The 24-cell is by definition the Voronoi region for the root lattice D_4 , and the reader may consult Coxeter [2] for more details.

Our proof of the main theorem above involves some new ideas that are quite different from Minkowski's proof for 1-tilings. We also prove the following counter-part to the main Theorem above.

Theorem 1.2. *Every rational polytope P that is centrally symmetric and has centrally symmetric facets must necessarily k -tile \mathbb{R}^d with a lattice, for some positive integer k .*

Moreover, the polytope P must k -tile \mathbb{R}^d with the rational lattice $\frac{1}{N}\mathbb{Z}^d$, where N is the lcm of the denominators of all the vertex coordinates of P .

The paper is organized as follows. Section 3 is devoted to the proof of the main result, namely Theorem 1.1, and comprises the main body of the paper. Section 4 is short, and is devoted to the proof of Theorem 1.2. In section 5 we provide a more analytic approach of the main result, using Fourier techniques. Although it is not crucial to supply another proof of the main result, this approach provides a Fourier lens through which we can view our results.

Kolountzakis has also studied this problem using the Fourier approach, and indeed our Fourier approach borrows some techniques from his work. In section 6 we give another necessary and sufficient condition for a polytope to k -tile \mathbb{R}^d , this time in terms of the solid angles of the vertices of P . Finally, in section 7 we mention some of the important open problems concerning polytopes that k -tile \mathbb{R}^d .

2 Definitions and preliminaries

We adopt the usual conventions and notation from combinatorial geometry. First, we recall that the **Minkowski sum** of two multisets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$ is the set $A + B = \{a + b : a \in A, b \in B\}$, and that the Minkowski difference is defined similarly by $A - B = \{a - b : a \in A, b \in B\}$.

For any set $A \subset \mathbb{R}^d$, its **opposite set** is defined as $-1 \cdot A = \{-a : a \in A\}$. We are particularly interested in the case that both A and B are polytopes. We are also keenly interested in the case that A is a polytope and B is a discrete set of vectors, so that here $A + B$ is a set of translated copies of the polytope A .

Given a convex body $P \subseteq \mathbb{R}^d$, ∂P denotes the boundary of P . The standard convention for ∂P includes the fact that it has (d -dimensional) Lebesgue measure 0, with respect to the Lebesgue measure of \mathbb{R}^d . We let the interior of a body P be denoted by $\text{Int}(P)$. Throughout the paper, Λ denotes an infinite discrete multiset of vectors in \mathbb{R}^d , which is not necessarily a lattice.

We say that body P **k -tiles** \mathbb{R}^d with the discrete multiset Λ , if after translating P by each vector $\lambda \in \Lambda$, almost every point of \mathbb{R}^d (except for the boundary points of translated copies of P) is covered by exactly k of these translated copies of P . This condition can be written more concisely as follows:

$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = k,$$

for all $v \notin \partial P + \Lambda$.

We also recall that a **facet** of a d -dimensional polytope is any one of its $(d - 1)$ -dimensional faces. We let $V^k(F)$ denote the k -dimensional volume of a k -dimensional object F , even if F resides in a higher dimensional ambient space, and sometimes we simply write $V(F)$ for the d -dimensional volume of a d -dimensional object $F \subset \mathbb{R}^d$. Finally, $\#(A)$ denotes the cardinality of any finite multiset A .

3 Proof of Theorem 1.1

To simplify the ensuing notation, we will assume that $-1 \cdot P$ k -tiles \mathbb{R}^d . We do not lose any generality, because $-1 \cdot P$ k -tiles \mathbb{R}^d if and only if P also k -tiles \mathbb{R}^d .

We say that $v \in \mathbb{R}^d$ is in *general position* if there are no points of Λ on the boundary of $P + v$. In other words, $v \notin \Lambda - \partial P$. We first prove the following elementary but useful lemma, giving an equivalent condition for k -tiling in terms of the number of Λ -points that lie in a 'typical' translate of P .

Lemma 3.1. *A convex polytope $-1 \cdot P$ k -tiles \mathbb{R}^d by translations with a multiset Λ if and only if $\#(\Lambda \cap \{P + v\}) = k$ for every v in general position.*

Proof. Suppose that $-1 \cdot P$ k -tiles \mathbb{R}^d . Then for every $v \notin \partial(-1 \cdot P) + \Lambda$ we can write

$$k = \sum_{\lambda \in \Lambda} 1_{\{-1 \cdot P + \lambda\}}(v) = \sum_{\lambda \in \Lambda} 1_{P+v}(\lambda) = \#(\Lambda \cap \{P + v\}).$$

It remains to mention that $\partial(-1 \cdot P) + \Lambda = \Lambda - \partial P$. The proof in the other direction is identical. \square

We need to introduce some useful and natural notation for the theorems that follow. Let \mathfrak{P} be the vector space of the real linear combinations of indicator functions of all convex polytopes in \mathbb{R}^d . Thus, for example, if P is any convex k -dimensional polytope and Q is any convex m -dimensional polytope, then $\frac{1}{3} \cdot 1_P - 2 \cdot 1_Q \in \mathfrak{P}$.

One of the most important operators for us is the following boundary operator, with respect to a vector $n \in \mathbb{R}^d$. It is the function $\partial_n : \mathfrak{P} \rightarrow \mathfrak{P}$, defined as follows:

$$\partial_n 1_P = 1_{F^+} - 1_{F^-},$$

where F^+ and F^- are the (possibly degenerate) facets of P with outward pointing normals n and $-n$, respectively. It is a standard vector space verification that this operation is also well-defined on \mathfrak{P} .

We also define this boundary operator on all of \mathfrak{P} , by letting it act as a linear operator on the linear combinations of indicator functions of polytopes. For example, another iteration of this operator on $P \subset \mathbb{R}^3$ yields $\partial_{n_2}(\partial_{n_1} P) = \partial_{n_2}(1_{F^+} - 1_{F^-}) = (1_{E_1} - 1_{E_2}) - (1_{E_3} - 1_{E_4})$, where E_1, E_2 are the edges (which are by definition the 1-dim'l faces) of F^+ , and E_3, E_4 are the edges of F^- . In this case, each of the four edges is orthogonal to both of the vectors n_1 and n_2 , as is seen in Figure 2 below.

For the sake of convenience, we also define the action of the boundary operator ∂_n on convex polytopes P as follows:

$$\partial_n P = \text{supp}(\partial_n 1_P) = \{v \in \mathbb{R}^d \mid \partial_n 1_P(v) \neq 0\},$$

so that the same symbol now acts on the subset P . However, we note that the more salient operator for our discussions is still $\partial_n 1_P$. It is useful to utilize both of these actions, the first being an action on indicator functions, and the second being an action on subsets of points $P \subset \mathbb{R}^d$.

We call a sequence $\mathbf{n} = (n_1, \dots, n_m)$ of vectors in \mathbb{R}^d an orthogonal frame if they are pairwise orthogonal to each other. We denote it by \mathbf{n}^\perp the subspace of \mathbb{R}^d consisting of those vectors which are orthogonal to every vector in the orthogonal frame \mathbf{n} .

We define $\partial_{\mathbf{n}} := \partial_{n_m} \dots \partial_{n_1}$, a composition of boundary operators that is read from right to left. In case $m = 0$, when an orthogonal frame \mathbf{n} is empty, we define $\partial_{\mathbf{n}}$ to be an identity operator. Similarly to $\partial_n P$ we define a boundary operator relative to a whole frame $\mathbf{n} = (n_1, \dots, n_m)$:

$$\partial_{\mathbf{n}} P = \text{supp}(\partial_{\mathbf{n}} 1_P) = \{v \in \mathbb{R}^d \mid \partial_{\mathbf{n}} 1_P(v) \neq 0\}.$$

Note that all the faces whose indicator functions appear in $\partial_{\mathbf{n}} 1_P$ must have codimension m , must be parallel to each other, and must have outward pointing normals n_m or $-n_m$ in $\partial_{n_{m-1}} \dots \partial_{n_1} P$.

We can now separate $\partial_{\mathbf{n}}$ into two parts: $\partial_{\mathbf{n}}^+$ and $\partial_{\mathbf{n}}^-$, corresponding to faces with outward normals n_m or $-n_m$, so that $\partial_{\mathbf{n}} 1_P = \partial_{\mathbf{n}}^+ 1_P - \partial_{\mathbf{n}}^- 1_P$. In other words, if $\partial_n 1_P = 1_{F^+} - 1_{F^-}$, then by definition $\partial_n^+ 1_P = 1_{F^+}$, and $\partial_n^- 1_P = 1_{F^-}$.

We say that $v \in \mathbb{R}^d$ is in general position w.r.t. the orthogonal frame \mathbf{n} , if there are no points of Λ on any boundary component of $\partial_{\mathbf{n}}(P + v)$. A more formal description which we will have occasion to use below is that $v \notin \Lambda - \partial \partial_{\mathbf{n}} P$.

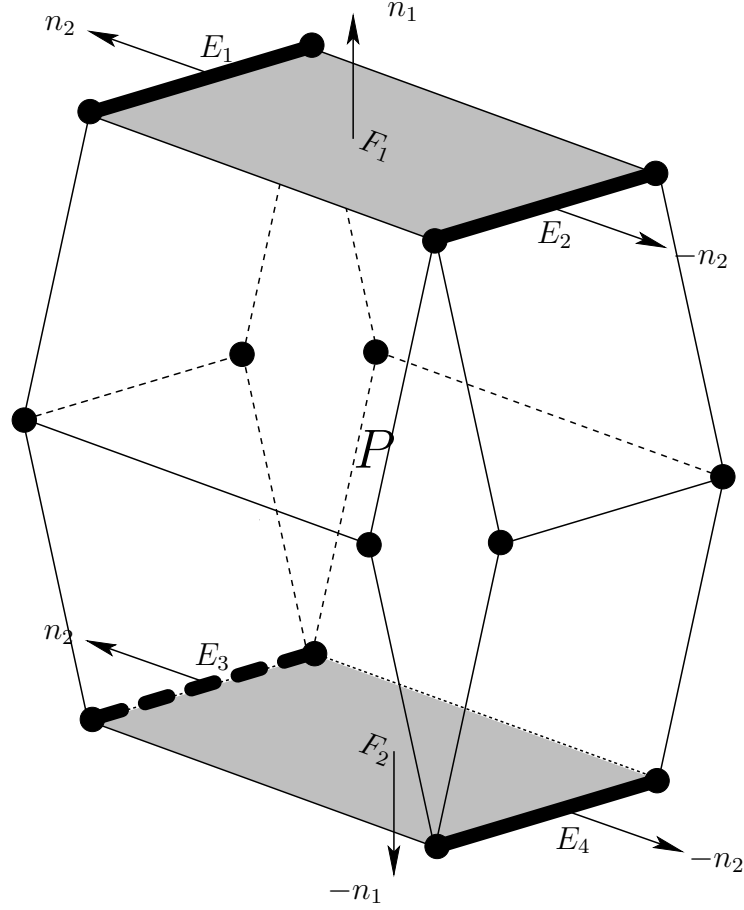


Figure 2: The boundary operator with respect to n_1 picks out the two facets F^+ and F^- , illustrating the definition of $\partial_n 1_P = 1_{F^+} - 1_{F^-}$. A second iteration of the boundary operator, this time with respect to n_2 , picks out the four edge vectors E_1, E_2, E_3 , and E_4 , thus visually illustrating the identity $\partial_{n_2}(\partial_{n_1} P) = \partial_{n_2}(1_{F^+} - 1_{F^-}) = (1_{E_1} - 1_{E_2}) - (1_{E_3} - 1_{E_4})$.

Even though we only need to consider orthogonal frames of size at most two in order to prove the theorem 1.1, we will prove two following lemmas in general case, for an orthogonal frame of any size.

Lemma 3.2. *Suppose $\#(\Lambda \cap \{P + v\}) = k$ for every v in general position. Let $\mathbf{n} = (n_1, \dots, n_m)$ be an orthogonal frame in \mathbb{R}^d . Then for any v in general position w.r.t \mathbf{n} the following formula holds:*

$$\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}} 1_{P+v}(\lambda) = 0. \quad (1)$$

Proof. We proceed by induction on m . We remark that for $m = 0$ the hypothesis tells us that $\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}} 1_{P+v}(\lambda) = k$, and for $m = 0$ this operator is by definition the identity operator. However, for each $m \geq 1$, we will show that $\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}} 1_{P+v}(\lambda) = 0$.

Suppose that for an $(m-1)$ -dimensional orthogonal frame $\mathbf{n}' = (n_1, \dots, n_{m-1})$ and for every v in general position w.r.t. \mathbf{n}' the formula holds:

$$\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}'} 1_{P+v}(\lambda) = \text{const}$$

Now consider any m -dimensional orthogonal frame $\mathbf{n} = (n_1, \dots, n_m)$, and v in general position w.r.t \mathbf{n} . We know that all Λ -points of $v + \partial_{\mathbf{n}} P$ lie in $v + \text{Int}(\partial_{\mathbf{n}} P)$. Therefore, one can pick sufficiently small ϵ' , such that no ϵ' -perturbation of v by a vector in \mathbf{n}^\perp removes or adds any Λ -points to $v + \partial_{\mathbf{n}} P$. Clearly, by doing so we do not change $\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}} 1_{P+v}(\lambda)$. On the other hand, we may choose an ϵ' -perturbation, $v_{\epsilon'}$, such that all Λ -points in $v_{\epsilon'} + \partial \partial_{\mathbf{n}'} P$ get either inside or outside of $v_{\epsilon'} + \partial_{\mathbf{n}'} P$ (see fig. 3).

Then consider two small perturbations of $v_{\epsilon'}$ in the directions n_m and $-n_m$: $v_{\epsilon'}^+ = v_{\epsilon'} + \epsilon n_m$ and $v_{\epsilon'}^- = v_{\epsilon'} - \epsilon n_m$, such that $v_{\epsilon'}^+$ and $v_{\epsilon'}^-$ are in general position w.r.t \mathbf{n}' , and ϵ small enough so that there are no points of Λ that lie in $P + v_{\epsilon'}^\pm$ and do not lie in $P + v_{\epsilon'}$ (such an ϵ can be found, because Λ is discrete).

By induction, $\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}^+}(\lambda) = \text{const} = \sum_{\lambda \in \Lambda} \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}^-}(\lambda)$.

On the other hand, recalling that by definition $\partial_{\mathbf{n}} P = \partial_{n_m} \partial_{\mathbf{n}'} P$,

$$\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}^+}(\lambda) - \sum_{\lambda \in \Lambda} \partial_{n_m}^+ \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}}(\lambda) = \sum_{\lambda \in \Lambda} \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}}(\lambda) \cdot 1_{\text{Int}(\partial_{\mathbf{n}'} P)}(\lambda) \quad (2)$$

$$= \sum_{\lambda \in \Lambda} \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}^-}(\lambda) - \sum_{\lambda \in \Lambda} \partial_{n_m}^- \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}}(\lambda). \quad (3)$$

It follows that $\sum_{\lambda \in \Lambda} \partial_{n_m}^+ \partial_{\mathbf{n}'} 1_{P+v}(\lambda) = \sum_{\lambda \in \Lambda} \partial_{n_m}^+ \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}}(\lambda) = \sum_{\lambda \in \Lambda} \partial_{n_m}^- \partial_{\mathbf{n}'} 1_{P+v_{\epsilon'}}(\lambda) = \sum_{\lambda \in \Lambda} \partial_{n_m}^- \partial_{\mathbf{n}'} 1_{P+v}(\lambda)$, which gives us (1), since $\partial_{\mathbf{n}} = \partial_{n_m}^+ \partial_{\mathbf{n}'} - \partial_{n_m}^- \partial_{\mathbf{n}'}$. \square

For any polytope in \mathbb{R}^d lying in an affine subspace parallel to \mathbf{n}^\perp , we may consider a naturally defined lower dimensional volume $\text{Vol}_{\mathbf{n}}$. For example, if $d = 3$ and $\mathbf{n} = (n_1, n_2)$, we get $\text{Vol}_{\mathbf{n}}$ to be just a length of a line segment in \mathbb{R}^3 . As we know $\partial_{\mathbf{n}} 1_P$ is a finite sum of indicator functions of polytopes lying in affine subspaces parallel to \mathbf{n}^\perp taken with $+$ or $-$ signs. For each such

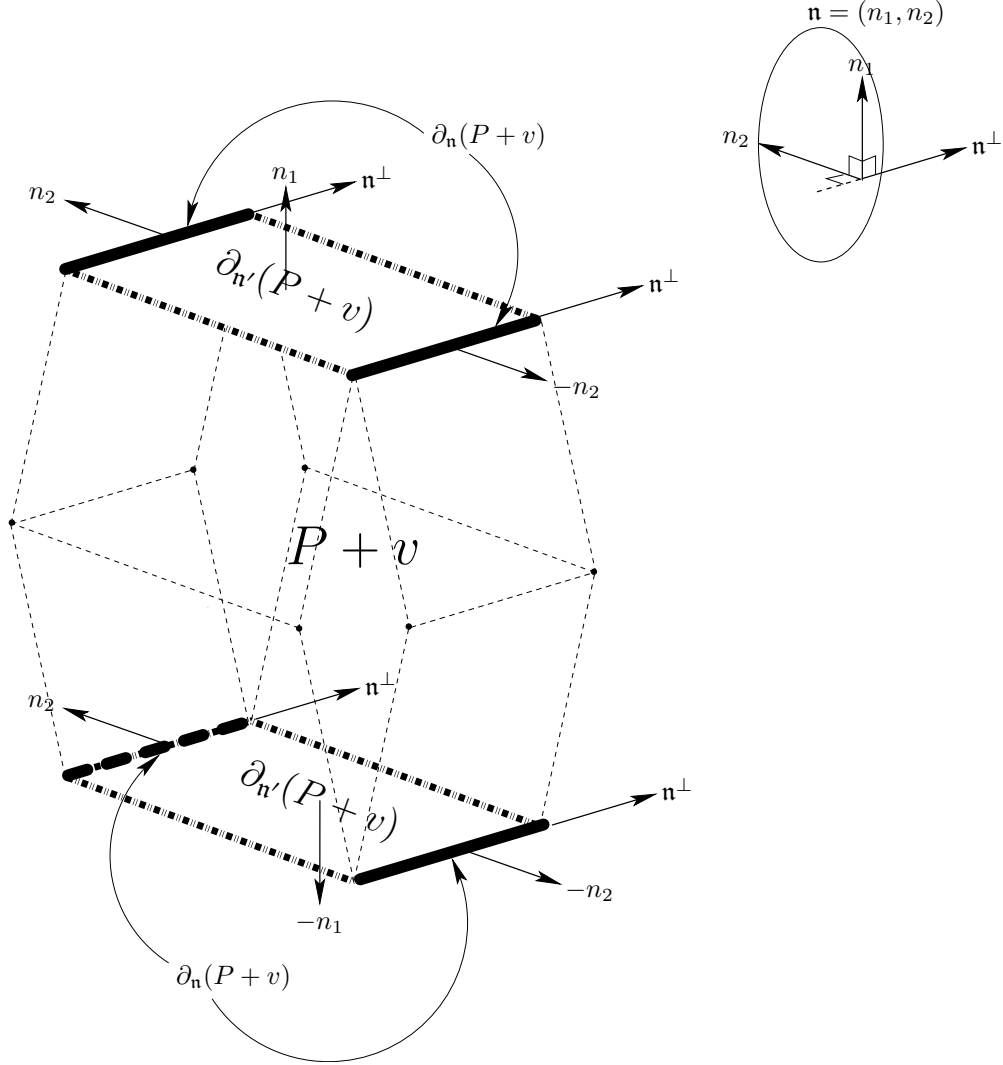


Figure 3: The ϵ' perturbation, along the n^\perp direction, insures that all Λ points have been removed from the four dotted edges on the upper facet and lower facet of $P+v$, giving us the set $\partial_{n'}(P+v)$. Also, the ϵ perturbation, along the n_2 direction, insures that all Λ points on the right-hand bold edges, attached to the normal vector $-n_2$, will end up outside of the perturbed set, and that all Λ points on the left-hand bold edges, attached to the normal vector n_2 , will end up inside the perturbed set.

indicator function 1_F let us denote by $\text{Vol}_{\mathbf{n}}(1_F)$ the volume of polytope F . Note that we can take any measurable object in the affine subspace parallel to \mathbf{n}^\perp instead of F .

We now extend the notion of $\text{Vol}(S)$ to a more general notion of a signed linear combination of volumes. We let $V_{\mathbf{n}}(\partial_{\mathbf{n}}1_P)$ denote the sum of the corresponding volumes taken with different signs, and in a similar way we can write $V_{\mathbf{n}}$ for any sum of positive and negative indicator functions. The next Lemma extends equality (1) in Lemma 3.2 from a discrete measure of facets to a continuous measure of facets.

Lemma 3.3. *Under the same assumptions of lemma 3.2, the following formula holds:*

$$V_{\mathbf{n}}(\partial_{\mathbf{n}}1_P) = 0.$$

Proof. Let us recall what we have so far. Lemma 3.2 tells us $\sum_{\lambda \in \Lambda} \partial_{\mathbf{n}}1_{P+v}(\lambda) = 0$ for any v with $v + \partial\partial_{\mathbf{n}}P$ containing no Λ points.

For each $\lambda \in \Lambda$ let us consider a set S of vectors v enjoying the property that $\lambda \in v + \partial_{\mathbf{n}}P$ and $\Lambda \cap \{v + \partial\partial_{\mathbf{n}}P\} = \emptyset$. We call the set S \mathbf{n} -interior w.r.t. λ . We can also realize the set S by excluding a finite number of lower dimensional polytopes (polytopes F with $V_{\mathbf{n}}(F) = 0$) from $\lambda - \partial_{\mathbf{n}}P$. We call a vector \mathbf{n} -internal if it belongs to \mathbf{n} -interior for some $\lambda \in \Lambda$.

Assume now that $V_{\mathbf{n}}(\partial_{\mathbf{n}}1_P) = A_1 \neq 0$. Let us also write $V_{\mathbf{n}}(|\partial_{\mathbf{n}}1_P|) = A_2 \geq |A_1| > 0$, where by $|\partial_{\mathbf{n}}1_P|$ we imply the sum of indicators of $\partial_{\mathbf{n}}1_P$ with all negative coefficients of indicators switched to their absolute value.

For any $R > 0$ we may consider a ball B_R in \mathbb{R}^d with the center at origin and given radius R . Clearly, there is a constant $C = C(P)$, such that $B_R + (-1)\partial_{\mathbf{n}}P \subset B_{R+C}$ (see fig. 4). For any positive real R we define $N(R) := \#\{B_R \cap \Lambda\}$. For each \mathbf{n} -internal $v \in B_R$ we may rewrite the formula from lemma 3.2 and get

$$\sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathbf{n}}1_{\lambda-P}(v) = 0.$$

This implies

$$V_{\mathbf{n}} \left(1_{B_R} \cdot \sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathbf{n}}1_{\lambda-P} \right) = 0.$$

Also we know that

$$\left| V_{\mathbf{n}} \sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathbf{n}}1_{\lambda-P} \right| = N(R+C)|A_1|.$$

$$\begin{aligned}
\left| V_{\mathbf{n}} \left(\sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathbf{n}} 1_{\lambda-P} \right) \right| &\leq \left| V_{\mathbf{n}} \left(1_{B_R} \cdot \sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathbf{n}} 1_{\lambda-P} \right) \right| + \\
&\quad \left| V_{\mathbf{n}} \left((1_{B_{R+2C}} - 1_{B_R}) \cdot \sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathbf{n}} 1_{\lambda-P} \right) \right| = \\
&\quad \left| V_{\mathbf{n}} \left((1_{B_{R+2C}} - 1_{B_R}) \cdot \sum_{\lambda \in (B_{R+C} \setminus B_{R-C}) \cap \Lambda} \partial_{\mathbf{n}} 1_{\lambda-P} \right) \right| \leq \\
&\leq V_{\mathbf{n}} \sum_{\lambda \in (B_{R+C} \setminus B_{R-C}) \cap \Lambda} |\partial_{\mathbf{n}} 1_{\lambda-P}| = A_2 \cdot (N(R+C) - N(R-C)).
\end{aligned}$$

Thus we get $(1 - \frac{|A_1|}{A_2})N(R+C) \geq N(R-C)$, which establishes an exponential growth of $N(R)$ in R . We can cover B_R by a disjoint union of $O(R^{2d})$ cubes whose side-length is $\frac{1}{R}$. Thus taking sufficiently large R we can find a cube K with side-length $\frac{1}{R}$, which contains more than k Λ -points. We can now translate P so that the cube K is contained in P , and therefore this translate of P now contains more than k Λ -points, a contradiction. \square

In order to finish the proof of main theorem, we need the following theorem by Minkowski [11].

Theorem 3.4 (Minkowski). *Convex polytope in \mathbb{R}^d with given facet normals and facet $(d-1)$ -volumes is unique up to translation.*

Proof of Theorem 1.1. We will first prove that P is centrally symmetric. Take any pair of facets of P , F^+ and F^- , with outward normals n and $-n$ respectively. Applying lemma 3.3 to $\mathbf{n} = (n)$ we get $V_{\mathbf{n}}(\partial_{\mathbf{n}} 1_P) = 0$, which means that $V(F^+) = V(F^-)$. Since n can be chosen arbitrarily, polytopes P and $(-1) \cdot P$ have equal codimension 1 volumes of facets in every direction. By theorem 3.4 we get that $P = (-1) \cdot P + v$ for some translation vector v , so P is centrally symmetric.

Similarly we prove that every facet of P is centrally symmetric. Given a pair of opposite facets F_1 and F_2 of P with outward normals n_1 and $-n_1$ respectively, consider any direction $n_2 \in (n_1)^\perp$ and two pairs of corresponding faces of codimension 2: F_1^+ and F_1^- are facets of F_1 with outward normals n_2 and $-n_2$ respectively, F_2^+ and F_2^- are facets of F_2 with outward normals n_2 and $-n_2$ respectively. Applying lemma 3.3 to $\mathbf{n} = (n_1, n_2)$ we get $V_{\mathbf{n}}(\partial_{\mathbf{n}} 1_P) = 0$, which means that $(V(F_1^+) - V(F_1^-)) - (V(F_2^+) - V(F_2^-)) = 0$. But since P is centrally symmetric, F_1^+ and F_2^- are symmetric to each other as well as F_1^- and F_2^+ , so $V(F_1^+) = V(F_2^-)$ and $V(F_1^-) = V(F_2^+)$.

Combining the last three equations we get an equality for codimension 2 faces of P : $V(F_1^+) = V(F_1^-)$. It follows that as $(d-1)$ -dimensional objects, F_1 and $(-1) \cdot F_1$, themselves have equal facets in every direction (in their affine span), and again by theorem 3.4 we get that F_1 is centrally symmetric. But since F_1 could be chosen arbitrarily among the facets of P , every facet of P is centrally symmetric, which concludes the proof of theorem 1.1. \square

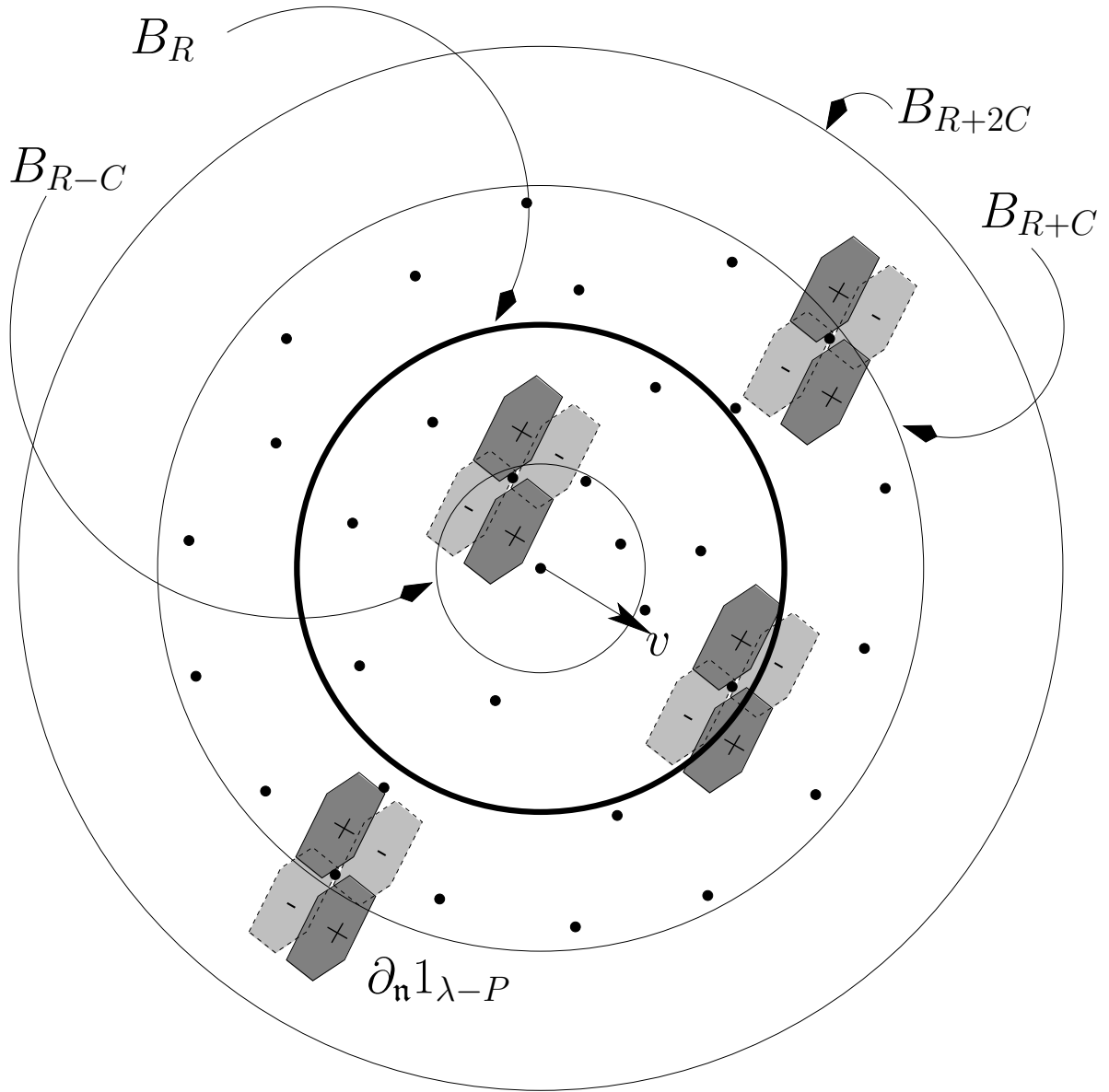


Figure 4: None of the Λ -translates of $\partial_n 1_{\lambda-P}$ can overlap more than two adjacent shells between the concentric balls.

Remark. We note that Lemma 3.2 gives us interesting information about the relationship between the Λ points that lie in various faces, for any frame that has more than 2 vectors. In contrast, Lemma 3.3 does not give us any additional information about the codimension 3 volumes (or higher codimension volumes). It is for this reason that we cannot conclude that codimension 3 faces of a k -tiling polytope are centrally symmetric, and in fact they are not in general centrally symmetric, as the example of the 24-cell shows.

4 Proof of Theorem 1.2

Proof. We may assume, without loss of generality, that our rational polytope P is an integer polytope, by dilating it by the lcm of the denominators of all of the rational coordinates of its vertices. Now, given that P has integer vertices, we will show that the polytope P k -tiles \mathbb{R}^d with $\Lambda = \mathbb{Z}^d$.

We claim that in every general position P has an equal number of integer points on every pair of opposite facets. Indeed, since it is centrally symmetric and has centrally symmetric facets (and integer vertices), any two opposite facets are translations of one another by some integer vector. It follows that for every integer point on a facet there is a corresponding integer point on an opposite facet, so their numbers are equal.

Now, consider any two general positions of P , say $P + u$ and $P + v$. There exists some path from u to v such that when we translate P along this path, no integer point of \mathbb{Z}^d collides with any co-dimension 2 face of the translates of P along this path (see fig. 5). But since in any general position the number of integer points on two opposite facets of P are equal, it follows that the number of points inside P along this path is constant. We conclude that any two general positions of P have the same number of interior integer points, say k . Thus, $-P$ k -tiles \mathbb{R}^d with the lattice \mathbb{Z}^d . \square

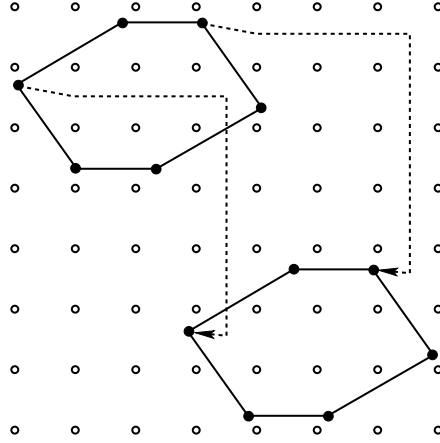


Figure 5: This polygon illustrates that fact that there is always a continuous path that a polygon P may take so that the vertices of P (and in general the codimension 2 faces of P) never pass through the discrete set of translations vectors Λ , shown here as a lattice.

5 An analytic approach, using Fourier techniques

In this section we give another proof of the main result, Theorem 1.1, but this time from the Fourier perspective, so that we may employ the language of generalized functions. The reader may consult the classic reference [12] for more information about Fourier analysis on Euclidean spaces. We begin once again with the definition of a k -tiling. Thus, we suppose that a polytope P k -tiles \mathbb{R}^d with some discrete multiset Λ . In other words, we assume that

$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = k,$$

for all $v \notin \partial P + \Lambda$. We can rewrite this condition as a convolution of generalized functions, as follows:

$$1_P * \delta_\Lambda = k, \tag{4}$$

where 1_P is the indicator function of P , and where $\delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$, where δ_λ is the unit point mass for the point $\lambda \in \Lambda$. That is, δ_λ equals 1 at the point λ and zero elsewhere. We first differentiate both sides of (4), with respect to any $\xi \in \mathbb{R}^d$, obtaining

$$\frac{d}{d\xi} (1_P * \delta_\Lambda) = \left(\frac{d}{d\xi} 1_P \right) * \delta_\Lambda = 0. \tag{5}$$

Next, we take the Fourier transform of both sides of (5), obtaining

$$(\xi \hat{1}_P) \hat{\delta}_\Lambda = 0, \tag{6}$$

where the last step uses the standard Fourier identities $\widehat{\left(\frac{d}{dx} F\right)}(\xi) = \xi \hat{F}(\xi)$, and $\widehat{F * G} = \hat{F} \hat{G}$. If we now have some more detailed knowledge about $\hat{1}_P$, then we can use (6) to proceed further.

The next result is a useful combinatorial version of Stokes' formula, which holds for the Fourier transform of the indicator function of any polytope. This is a result about $\hat{1}_P$ that appears to be not as well-known, so we prove it in complete detail. For the transform of a function on \mathbb{R}^d , we use the standard definition:

$$\hat{1}_P(\xi) := \int_P \exp(2\pi i \langle \xi, x \rangle) dx,$$

valid for any $\xi \in \mathbb{R}^d$, because P is compact.

Theorem 5.1. *Let F be a k -dimensional polytope in \mathbb{R}^d , for any $k \leq d$. Let $Proj_F(\xi)$ denote the orthogonal projection of ξ onto the k -dimensional subspace of \mathbb{R}^d that is parallel to F . Moreover, for each $(k-1)$ -dimensional face $G \in \partial F$, let n_G be its outward pointing normal vector. Then the Fourier transform of the indicator function of F can be written as follows:*

Case I. If $Proj_F(\xi) = 0$, then

$$\hat{1}_F(\xi) = V^k(F) \exp(2\pi i \Phi),$$

where Φ is the constant value of the function $\phi(x) := \langle \xi, x \rangle$ on F .

Case II. If $Proj_F(\xi) \neq 0$, then

$$\hat{1}_F(\xi) = -\frac{1}{2\pi i} \sum_{G \in \partial F} \frac{\langle Proj_F(\xi), n_G \rangle}{\|Proj_F(\xi)\|^2} \hat{1}_G(\xi).$$

Proof. We note that the gradient of $\phi(x) := \langle \xi, x \rangle$, with respect to the Riemannian structure of the submanifold $F \subset \mathbb{R}^d$, is simply the projection of the d -dimensional Euclidean gradient of ϕ onto F . We denote this projection by $\text{grad}_F \phi$ in the argument that follows. Fix any $\xi \in \mathbb{R}^d$.

Case I. If $Proj_F(\xi) = 0$, then $\text{grad}_F \phi = 0$, so that ϕ is constant on F . The Fourier integral defining $\hat{1}_F(\xi)$ in this case degenerates into an integral of a constant function on F , hence the conclusion of the theorem for this case.

Case II. If $Proj_F(\xi) \neq 0$, then from the linearity of ϕ it follows that $\text{grad}_F \phi(x) = 2\pi Proj_F(\xi)$ is a constant vector field on F . The identity

$$\text{div}_F \text{grad}_F \exp(2\pi i \phi(x)) = (2\pi i)^2 \|\text{grad}_F \phi(x)\|^2 \exp(2\pi i \phi(x))$$

shows us that $\exp(2\pi i \phi(x))$ is an eigenfunction of the Laplacian, with eigenvalue

$$\lambda := (2\pi i)^2 \|\text{grad}_F \phi\|^2 \neq 0.$$

Hence

$$\begin{aligned} \hat{1}_F(\xi) &= \int_F e^{2\pi i \phi(x)} \\ &= \frac{1}{\lambda} \int_F \text{div}(\text{grad}_F e^{2\pi i \phi(x)}) dF \\ &= \frac{1}{\lambda} \sum_{G \in \partial F} \int_G \langle \text{grad}_F e^{2\pi i \phi(x)}, n_G \rangle dG, \end{aligned}$$

where we've used the identity for the Laplacian above in the second equality, and Stokes' theorem for the polytope F and its finite collection of boundary polytope components $G \in \partial F$ in the third equality. Unravelling the remaining definitions, we get:

$$\begin{aligned}\hat{1}_F(\xi) &= \frac{2\pi i}{\lambda} \sum_{G \in \partial F} \langle \text{grad} \phi(x), n_G \rangle \int_G e^{2\pi i \phi(x)} dG \\ &= -\frac{1}{2\pi i} \sum_{G \in \partial F} \frac{\langle \text{Proj}_F(\xi), n_G \rangle}{\|\text{Proj}_F(\xi)\|^2} \hat{1}_G(\xi).\end{aligned}$$

□

The result above uses functions, as opposed to generalized functions, but we may indeed pass to generalized functions, abusing the notation $\hat{1}_P$ only slightly.

Applying Theorem (5.1) above to the generalized function 1_P , we may continue from (6) to get the identity

$$\left(\sum_{F \in \partial P} \xi \frac{\langle \xi, n_F \rangle}{\langle \xi, \xi \rangle} \hat{1}_F \right) \hat{\delta}_\Lambda = 0, \quad (7)$$

valid for any nonzero $\xi \in \mathbb{R}^d$. We also note that the sum runs over all the (codimension 1) facets F of the boundary ∂P . It now follows, upon taking the inner product with ξ , that

$$\left(\sum_{F \in \partial P} \langle \xi, n_F \rangle \hat{1}_F \right) \hat{\delta}_\Lambda = 0. \quad (8)$$

Taking Fourier transforms again, we may rewrite the last equation as

$$\left(\sum_{F \in \partial P} \left(\frac{d}{dn_F} \right) (1_F) \right) * \delta_\Lambda = 0. \quad (9)$$

We now focus our attention on each *pair* of facets of P , as in the first section. Thus, we consider a facet F^+ with its outward pointing normal $n(F)$, and a parallel facet F^- , with its outward pointing normal $-n(F)$.

Lemma 5.2. *For each facet F of P , we have the identity*

$$(1_{F^+} - 1_{F^-}) * \delta_\Lambda = 0.$$

Proof. We assume that $(1_{F^+} - 1_{F^-}) * \delta_\Lambda \neq 0$. Therefore there exists a small ball B_r , of radius r , such that for any nonnegative, nonzero test function f whose support is contained in B_r , we have $\langle (1_{F^+} - 1_{F^-}) * \delta_\Lambda, f \rangle \neq 0$. We may further assume that the support of f is disjoint from the support of $(1_{G^+} - 1_{G^-}) * \delta_\Lambda$, for any facet G of P where $G \neq F$. Indeed, the discreteness of Λ guarantees that we can find such a ball B_r on which f satisfies the above conditions.

Now we construct a test function g whose support is contained in B_r , with positive derivative $\left(\frac{d}{dn_F} \right)$ along the direction n_F , in a small ϵ vicinity of $B_r \cap \text{Supp}((1_{F^+} - 1_{F^-}) * \delta_\Lambda) := D_\epsilon$. To construct such a g , we first restrict f to D_ϵ , call it f_0 . We now multiply f_0 by a one dimensional

smooth bump function b whose derivative on $[-\epsilon, \epsilon]$ is positive, and whose support lives in $[-2\epsilon, 2\epsilon]$. Thus $g := f_0 \cdot b$ has positive derivative on D_ϵ . When we insert this g into (9), we arrive at a contradiction. Indeed $\langle (1_{G^+} - 1_{G^-}) * \delta_\Lambda, \frac{d}{dn_G} g \rangle = 0$ for $G \neq F$ because the choice of the support of g . On the other hand $\langle (1_{F^+} - 1_{F^-}) * \delta_\Lambda, \frac{d}{dn_F} g \rangle \neq 0$ by the construction, since $\frac{d}{dn_F} g$ is positive in the vicinity of the support of $(1_{F^+} - 1_{F^-}) * \delta_\Lambda$. \square

We finish this section by remarking that iteration of Lemma 5.2 allows us to establish the same conclusion as Lemma 3.2. The next iteration would be applied to a normal vector to a facet of F^+ within the affine span of the facet F^+ . The Fourier analogue of Lemma 3.3 involves the scalar product of $(1_{F^+} - 1_{F^-}) * \delta_\Lambda$ against an “approximate identity” function, compactly supported on a large ball. The main Theorem 1.1 now follows in a similar manner as in the previous section.

6 Another equivalent condition for k -tiling, using solid angles

Here we show that it is possible to reinterpret the condition that a polytope k -tiles \mathbb{R}^d by considering all of the solid angles $\omega_P(\lambda)$ of the d -dimensional convex polytope P , at each point $\lambda \in \Lambda$. For any point $\lambda \in \mathbb{R}^d$, we define the solid angle at λ to be the proportion of a small sphere of radius R , centered at λ , which intersects P . More precisely, the solid angle is defined by

$$\omega_P(\lambda) = \lim_{R \rightarrow 0} \frac{V(\{\lambda + B_R\} \cap P)}{V(B_R)},$$

where $V(S)$ is the d -dimensional volume of S . The following Theorem is of independent interest, showing another interesting equivalent condition for k -tiling Euclidean space.

Theorem 6.1. *A polytope P k -tiles \mathbb{R}^d with the multiset Λ if and only if*

$$\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda) = k,$$

for every $v \in \mathbb{R}^d$.

Proof. Suppose that P k -tiles \mathbb{R}^d with the multiset Λ . We know from Theorem 1.1 that P must be centrally symmetric, and therefore $-P$ k -tiles as well, with the multiset Λ . By Lemma 3.1 $\#(\Lambda \cap \{P + x\}) = k$ for almost every $x \in \mathbb{R}^d$. We can therefore integrate this equality in the variable x , over a d -dimensional ball $B_R(v)$ with center in v and radius R , as follows:

$$\begin{aligned}
k \cdot V(B_R(v)) &= \int_{B_R(v)} k \, dx = \int_{B_R(v)} \#(\Lambda \cap \{P + x\}) \, dx \\
&= \int_{B_R(v)} \sum_{\lambda \in \Lambda} 1_{\lambda - P}(x) \, dx \\
&= \sum_{\lambda \in \Lambda} \int_{B_R(v)} 1_{\lambda - P}(x) \, dx \\
&= \sum_{\lambda \in \Lambda} V(B_R(v) \cap \{\lambda - P\}) \\
&= \sum_{\lambda \in \Lambda} V(\{\lambda - B_R\} \cap \{P + v\})
\end{aligned}$$

It follows that $k = \sum_{\lambda \in \Lambda} \frac{V(\{\lambda - B_R\} \cap \{P + v\})}{V(B_R(v))}$, which approaches $\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda)$ as R goes to 0.

In the other direction, the assumption that $\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda) = k$ is, in general position, equivalent to the statement that $\#(\Lambda \cap \{P + x\}) = k$. By Lemma 3.1 we conclude that $-P$ k -tiles with the multiset Λ . Finally, by Theorem 1.1 we know that P is centrally symmetric, so that P k -tiles with the same multiset Λ . \square

We note that a particularly interesting choice of v in this Theorem is the value $v = 0$, so that we can in fact have points in Λ coincide with vertices of P . This equivalent condition allows us to consider such coincidences without having to translate P into general position.

7 Some open questions

We conclude our paper with some fascinating open questions which the main results of the present paper suggest as a natural research direction for k -tilings, a relatively new area.

1. Recall that the Venkov-McMullen condition for the existence of belts consisting of 4 or 6 parallel codimension 2 faces allowed an “if and only if” characterization for 1-tiling polytopes. Find the analogous additional condition that would give a complete characterization for k -tiling polytopes.
2. Classify the combinatorial types of all polytopes which k -tile \mathbb{R}^d by translations.

We note that for the classical question of 1-tiling \mathbb{R}^d by parallelotopes (and parallelotopes are the only objects that can tile \mathbb{R}^d , by McMullen’s theorem), there are exactly 5 combinatorially distinct parallelotopes in \mathbb{R}^3 , and exactly 52 distinct parallelotopes in \mathbb{R}^4 . It is still not known how many combinatorially distinct parallelotopes there are in dimensions 5 and higher. It is also not known how many facets a parallelotope may have in general (see [4] for references).

3. Prove or disprove that if any polytope k -tiles \mathbb{R}^d by translations, then it also m -tiles \mathbb{R}^d by a lattice, for a possibly different m .

This would give an analogue of the McMullen Theorem for 1-tiling parallelotopes in \mathbb{R}^d , but appears to be a very difficult problem.

4. Prove or disprove that if a 3-dimensional polytope, which is not a prism, k -tiles \mathbb{R}^3 by translations with a multiset Λ , then Λ is a union of a finite number of 3-dimensional lattices.

This would prove the 3-dimensional analogue of Kolountzakis' 2-dimensional result [5].

5. Is it always true that whenever P k -tiles with a multiset Λ , it follows that $\Lambda + v = \Lambda$ for some $v \in \mathbb{R}^2$? (This is one of Kolountzakis' open questions in [6])

6. Find, or estimate, the smallest k for which a given polytope can k -tile \mathbb{R}^d . This problem is open even in two dimensions.

References

- [1] A.D.Alexandrov, *Convex polyhedra*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad (1950), Akademie-Verlag, Berlin (1958), reprinted in English, by Springer-Verlag, Berlin (2005).
- [2] H.M.S.Coxeter, *Regular polytopes*, 2nd ed., MacMillan, New York (1963), 3rd ed., Dover, New York (1973).
- [3] E.S.Fedorov, *The symmetry of regular systems of figures*, Zap. Miner. Obshch. 21 (1885), pp. 1-279. (In Russian.)
- [4] P.M. Gruber, *Convex and discrete geometry*, Springer, Berlin, (2007), pp. 1-589.
- [5] M.N.Kolountzakis, *On the structure of multiple translational tilings by polygonal regions*, Discrete Comput. Geom. **23** (2000), 4, pp. 537-553.
- [6] M.N.Kolountzakis, *The study of translational tiling with Fourier Analysis*, Lectures given at the Workshop on Fourier Analysis and Convexity, Universita di Milano-Bicocca, June 11-22, (2001).
- [7] M.N.Kolountzakis and M.Matolcsi, *Tilings by translation*, La Gaceta de la Real Sociedad Espanola, **13** (2010).
- [8] J.C. Lagarias and Y.Wang, *Tiling the line with translates of one tile*, Inventiones math. **124** (1996), pp. 341-365.
- [9] P.McMullen, *Convex bodies which tile space by translation*, Mathematika **27** (1980), pp. 113-121.
- [10] P.McMullen, *Space tiling zonotopes*, Mathematika, **22** (1975), pp. 202-211.
- [11] H.Minkowski, *Allgemeine Lehrsätze über die convexen Polyeder*, Nachr. Ges. Wiss. Göttingen. 1897, pp. 198-219.

- [12] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, No. 32 (1971), pp. 1-297.
- [13] B. A. Venkov, *On a class of Euclidean polyhedra*, Vestnik Leningrad Univ. Ser. Math. Fiz. Him. **9** (1954), pp. 11-31 (Russian).
- [14] U. Bolle, *On multiple tiles in \mathbb{R}^2* , Intuitive Geometry (1994).
- [15] G. Ziegler, *Lectures on Polytopes*, Springer Graduate Texts in Mathematics (1995), pp. 1-365.